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SEMISIMPLE HOPF ALGEBRAS OF DIMENSION 6, 8

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ABSTRACT

We determine the isomorphic classes of 6 or 8 dimensional semisimple Hopf algebras A over an algebraically closed field such that $(\dim A)1 \neq 0$.

Introduction

In [LR] Larson and Radford prove that a semisimple Hopf algebra A of odd dimension ≤ 19 over a field k is commutative and cocommutative, so that, if k is algebraically closed, A is isomorphic to the dual k^G of the group-like Hopf algebra kG of an abelian group G. As easily seen, the conclusion holds true, if the dimension $\dim A$ is either 2 or 4. In this paper we determine the isomorphic classes of semisimple Hopf algebras A of dimension 6, 8 over an algebraically closed field k such that $(\dim A)1 \neq 0$. As a conclusion, such a Hopf algebra of dimension 6 is either commutative or cocommutative, while there is only one (up to isomorphisms) semisimple Hopf algebra of dimension 8 that is neither commutative nor cocommutative. To show this, an important role is played by the biproduct $B \times H$ [R] and the bicrossed product $K \bowtie H$ [H1, 2].

We work over a field k. Everything takes place over k. In particular $\otimes$ means $\otimes_k$. Let A be a finite dimensional Hopf algebra. The coalgebra structure and the antipode of A is written, as usual, by Δ, ε, S or $\Delta_A, \varepsilon_A, S_A$. A^* denotes the Hopf algebra $\mathrm{Hom}_k(A, k)$ of linear dual. If G is a finite group, kG denotes the group-like Hopf algebra of G and k^G means $(kG)^*$. C_n denotes the cyclic group of order n.

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1. Semisimple Hopf algebras of dimension 6

Throughout this section we suppose k is an algebraically closed field of characteristic $\neq 2, 3$.

Let A be a semisimple Hopf algebra of dimension 6.

LEMMA 1.1:

(1) *As an algebra A is isomorphic to either $k \times \cdots \times k$ (6 times) or $k \times k \times M_2(k)$, where $M_2(k)$ denotes the algebra of 2×2 matrices.*

(2) *A is cosemisimple and involutory.*

Proof: (1) This follows by counting dimensions.

(2) By (1) there exists a 2-dimensional group-like Hopf subalgebra of A^*. Hence Part (2) follows by applying [LR, Thms. 2.9, 2.11] to A^*. ∎

PROPOSITION 1.2: *In A there exist a 2-dimensional group-like Hopf subalgebra H and a 3-dimensional left coideal subalgebra B such that the inclusion $H \hookrightarrow A$ splits as a Hopf algebra map and that*

$$\mu: B \otimes H \to A, \quad \mu(b \otimes h) = bh$$

is an isomorphism.

Proof: By (1.1.1) there is a Hopf algebra quotient $\pi: A \to J$ such that $\dim J = 2$. Let

$$B = \{a \in A |\ (1 \otimes \pi) \circ \Delta(a) = a \otimes \pi(1)\},$$

the left coideal subalgebra of right J-coinvariants. By dualizing [MD, Thm. 3.5] we have that there is a right J-colinear map $\phi: J \to A$ such that

$$B \otimes J \to A, \quad b \otimes c \mapsto b\phi(c)$$

is an isomorphism. Hence $\dim B = 3$. If we show π has a Hopf algebra section $\imath$, then the proof is completed by setting $H = \imath(J)$, since $\imath$ can be chosen as ϕ.

We apply (1.1.1) to A^* to see that there is a 2-dimensional group-like Hopf subalgebra $H \subset A$. If $H \subset B$, then by the Nichols–Zoeller Theorem [NZ, Thm. 7] B would be H-free, so $\dim H$ would divide $\dim B$, a contradiction. Hence $H \not\subset B$ and $H \cong J$ via π. This means that π has a required section. ∎

Let H, B be as in (1.2). By [R, part of Thm. 3(b)], B is a left H-module algebra with action

$$h \rightharpoonup b = \sum h_{(1)} b S(h_{(2)}) \quad (h \in H, b \in B),$$

where $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$. We regard H (resp. B) as a quotient Hopf algebra (resp. a quotient coalgebra) of A via the isomorphism

$$H \cong A/B^+A \quad \text{(resp. } B \cong A/AH^+)$$

induced from μ, where $B^+ = \mathrm{Ker}(\varepsilon_A|_B), H^+ = \mathrm{Ker}\ \varepsilon_H$. We remark B is not necessarily a bialgebra. By [R, part of Thm. 3(b)], B is a left H-comodule coalgebra with coaction

$$\lambda\colon B \to H \otimes B, \quad \lambda(b) = (\pi \otimes 1) \circ \Delta_A(b),$$

where $\pi\colon A \to H$ is the quotient map. We write

$$\lambda(b) = \sum b_H \otimes b_B \quad (b \in B).$$

Following Radford we write by $B \times H$ the vector space $B \otimes H$ given an algebra structure of smash product constructed from $(H, B, \rightharpoonup)$ and a coalgebra structure of smash coproduct constructed from (H, B, λ). Thus $B \times H$ has the following multiplication and comultiplication:

$$(b \times h)(b' \times h') = \sum b(h_{(1)} \rightharpoonup b') \times h_{(2)}h',$$
$$\Delta(b \times h) = \sum (b_{(1)} \times b_{(2)H}h_{(1)}) \otimes (b_{(2)B} \times h_{(2)}),$$

where $b, b' \in B, h, h' \in H$.

LEMMA 1.3 ([R, Thm. 3, Prop. 2(b)]):

(1) *$B \times H$ is a Hopf algebra and $\mu\colon B \times H \to A$ is a Hopf algebra isomorphism.*

(2) *B has a convolution-inverse $S_B\colon B \to B$ of the identity map.*

(3) *The antipode S of $B \times H$ is given by*

$$S(b \times h) = \sum (1 \times S_H(b_H h))(S_B(b_B) \times 1),$$

where $b \in B, h \in H$.

LEMMA 1.4:

(1) *The algebra B is semisimple, so $B \cong k \times k \times k$.*

(2) *The coalgebra B is cosemisimple. Hence B is spanned by group-likes.*

Proof: (1) By [M, Prop. 2.10(1)] B is a Frobenius algebra. Hence Part (1) follows from the equivalence (1) $\Leftrightarrow$ (7) in [M, Thm. 2.1].

(2) Apply Part (1) to A^*. ∎

By (1.2), (1.4.2), we can write

$$B = k1 \oplus kx_+ \oplus kx_-, \quad H = k1 \oplus kz,$$

where $x_\pm, z$ are group-likes. Note that by (1.3.2) $x_\pm$ are units in B and $S_B(x_\pm) = x_\mp^{-1}$. We identify $H = H^*$ via the unique Hopf algebra isomorphism, which is induced from the pairing

$$\langle z^i, z^j \rangle = (-1)^{ij} \quad (i, j = 0, 1)$$

of the cyclic group $< z >$ of order 2. Hence the dual basis e_i of z^i is given by

$$e_i = \frac{1}{2}(1 + (-1)^i z) \quad (i = 0, 1). \tag{1.5}$$

Lemma 1.6: *Suppose A is neither commutative nor cocommutative.*

(1) $z \rightharpoonup 1 = 1, z \rightharpoonup x_\pm = x_\mp$.

(2) $\lambda(1) = 1 \otimes 1, \quad \lambda(x_\pm) = e_0 \otimes x_\pm + e_1 \otimes x_\mp$.

(3) $x_+^2 = x_-^2$.

Proof: $\rightharpoonup, \lambda$ must not be trivial since, if $\rightharpoonup$ (resp. λ) is trivial, A is commutative (resp. cocommutative).

(1) This follows since from the definition of $\rightharpoonup$ the action of z is a (non-trivial) coalgebra automorphism fixing 1. See also [R, Thm. 1].

(2) Since λ is a unitary comeasuring, by the dual action any group-like in H^* acts on B as a coalgebra automorphism fixing 1. Hence Part (2) follows.

(3) Since A is involutory by (1.1.2), one has

$$\sum S(a_{(2)})a_{(1)} = \varepsilon(a)1 \quad (a \in A). \tag{1.7}$$

Take $a = b \times 1$ ($b \in B$) and compute the left-hand side in $B \times H$ using (1.3.3). Then, writing $\Delta_B(b) = \Sigma b_{(1)} \otimes b_{(2)}$ ($b \in B$), the comultiplication of B, one has

$$\begin{aligned}\sum S(b_{(2)B} \times 1)(b_{(1)} \times b_{(2)H}) &= \sum (1 \times S_H(b_{(2)BH}))(S_B(b_{(2)BB})b_{(1)} \times b_{(2)H}) \\ &= \sum (1 \times S_H(b_{(2)H(2)}))(S_B(b_{(2)B})b_{(1)} \times b_{(2)H(1)}) \\ &= (\sum S_H(b_{(2)H}) \rightharpoonup S_B(b_{(2)B})b_{(1)}) \times 1 \text{ (since } S_H^2 = 1).\end{aligned}$$

Hence by (1.7) it follows that

$$\sum b_{(2)H} \rightharpoonup S_B(b_{(2)B})b_{(1)} = \varepsilon(b)1 \quad (b \in B),$$

since $S_H = 1$. Set $b = x_+$, then one has

$$1 + \frac{1}{2}(x_-^{-1}x_+ - x_+^{-1}x_-) = 1,$$

so $x_+^2 = x_-^2$. ∎

PROPOSITION 1.8: *A is either commutative or cocommutative.*

Proof: We suppose A is neither commutative nor cocommutative to show a contradiction.

We have $B = k \times k \times k$ by (1.4.1). Let $e = (1, 0, 0)$ be the unique primitive idempotent such that $\varepsilon(e) = 1$. Then $x_+ = (1, c, c')$, where $c, c' \in k$ are non-zero. Since the action of z on B is an algebra automorphism fixing e, one has $x_- = z \rightharpoonup x_+ = (1, c', c)$. By (1.6.3) it follows that $c^2 = c'^2$. This implies that $c = -c'$, since it cannot happen that $c = c'$ for $x_+ \neq x_-$. By definition of Δ_B or [R, Thm. 1 (b)], one has

$$\Delta_B(bb') = \sum b_{(1)}(b_{(2)H} \rightharpoonup b'_{(1)}) \otimes b_{(2)B}b'_{(2)} \tag{1.9}$$

for $b, b' \in B$. We set $b = b' = x_+$ and compute each side. Write $e' = 1 - e = (0, 1, 1)$, $t = c^2$. Then the right-had side of (1.9) equals

$$\begin{aligned} x_+(e_0 \rightharpoonup x_+) &\otimes x_+x_+ + x_+(e_1 \rightharpoonup x_+) \otimes x_-x_+ \\ &= e \otimes (e + te') + te' \otimes (e - te') \\ &= e \otimes e + te \otimes e' + te' \otimes e - t^2 e' \otimes e'. \end{aligned}$$

On the other hand, one sees

$$x_+^2 = t1 + \frac{1-t}{2}x_+ + \frac{1-t}{2}x_-,$$

so that

$$\Delta_B(x_+^2) = t1 \otimes 1 + \frac{1-t}{2}x_+ \otimes x_+ + \frac{1-t}{2}x_- \otimes x_-.$$

Multiply this by $e' \otimes e'$, then one has

$$\Delta_B(x_+^2)(e' \otimes e') = te' \otimes e' + t(1-t)f \otimes f,$$

where $f = (0, 1, -1)$. Hence for (1.9) it must hold that

$$t = -t^2, \quad t(1-t) = 0.$$

This has no non-zero solution. Thus a contradiction is shown. ∎

THEOREM 1.10: *6-Dimensional semisimple Hopf algebras over an algebraically closed field k of characteristic $\neq 2, 3$ consist of 3 isomorphic classes, which are represented by*

$$kC_6, \quad k\mathfrak{S}_3, \quad k^{\mathfrak{S}_3},$$

where $\mathfrak{S}_3$ denotes the symmetric group of degree 3.

Proof: These are not isomorphic to each other, as seen easily. These represent all isomorphic classes by (1.8) and since $kC_6 \cong k^{C_6}$. ∎

2. Semisimple Hopf algebras of dimension 8

In this section, except in (2.14), we suppose k is an algebraically closed field of characteristic $\neq 2$.

Let A be a semisimple Hopf algebra of dimension 8.

LEMMA 2.1:

(1) *As an algebra A is isomorphic to either $k \times \cdots \times k$ (8 times) or $k \times k \times k \times k \times M_2(k)$.*

(2) *A is cosemisimple.*

Proof: Similar to the proof of (1.1). ∎

By (2.1.2) A is a group-like Hopf algebra if it is cocommutative, and is the dual of such a Hopf algebra if commutative.

We suppose in addition A is neither commutative nor cocommutative. Let $G = G(A)$, the group-likes in A. Write $K = kG$ and $H = A/K^+A$, where $K^+ = \mathrm{Ker}\, \varepsilon_K$.

LEMMA 2.2:

(1) *G has order 4.*

(2) *The Hopf subalgebra $K \subset A$ is normal, that is, $K^+A = AK^+$. Hence H is a quotient Hopf algebra.*

(3) *$H \cong kC_2$.*

(4) *There is a unitary, co-unitary, left K-linear and right H-colinear isomorphism*

$$\alpha: K \otimes H \cong A.$$

Proof:

(1) Apply (2.1.1) to A^*.

(4) This follows by [Sch, Thm. 2.4 (2)] and [MD, Remark 1.8]. See also [MD, Prop. 3.2 (1)].

(2) It follows by (4) that H is a 2-dimensional quotient right A-module of the semisimple algebra A. Hence H is a direct sum of $H^+ = \mathrm{Ker}\ \varepsilon_H$ and some (two-sided) 1-dimensional ideal of A. This implies that H is a quotient algebra of A.

(3) Since by (2.1.2) A is cosemisimple, so is H (see the proof of (1.4.2)). Hence Part (3) follows. ∎

The isomorphism α gives to $K \otimes H$ a Hopf algebra structure of **bicrossed product** [H1, Kapitel 5; H2, Sect. 3]. We write this Hopf algebra by $K \bowtie H$. Thus $K \bowtie H$ is as an algebra a crossed product of H over K with some data

(action) $\rightharpoonup: H \otimes K \to K$

(cocycle) $\sigma: H \otimes H \to K,$

and as a coalgebra a crossed coproduct of K over H with some data

(coaction) $\rho: H \to H \otimes K$

(dual cocycle) $\theta: H \to K \otimes K.$

(In [H2] $K \bowtie H$ is denoted alternatively by $K\#_{(\sigma,\theta)}H$.) We remark $\rightharpoonup$ is co-unitary, ρ unitary and σ normalized, since α is unitary and co-unitary. (We need not know all of the compatible conditions of these data. We have only to know at most the conditions i, ii in [P, Lemma 1.1] for group crossed products.)

To find all A, which are neither commutative nor cocommutative as supposed, first we determine $\rightharpoonup, \rho$. Note $\rightharpoonup, \rho$ are independent of choice of α (see [Si, Prop. 2.9] in the graded case, and [H1, Satz 3.1.20; H2, Prop. 3.11] in the non-graded case). By (2.23) H is written as

$$H = k1 \oplus kz,$$

where z is a group-like.

PROPOSITION 2.3:

(1) *ρ is trivial.*

(2) *$G \cong C_2 \times C_2$.*

(3) *The action $\rightharpoonup$ is determined by*

$$z \rightharpoonup 1 = 1, \qquad z \rightharpoonup x = y, \qquad z \rightharpoonup y = x, \qquad z \rightharpoonup xy = xy,$$

where $x, y \in K$ are group-likes $\neq 1$.

Proof:

(1) Note $K^* \cong kG$, since G is abelian. Since ρ is a unitary comeasuring, $\rho^*\colon k^{C_2} \otimes kG \to k^{C_2}$ is a counitary measuring, so ρ^* is trivial. Hence ρ is trivial.

(2) By (1) the algebra A^* ($\cong K^* \bowtie H^*$) is isomorphic to the twisted group ring $H^{*t}[G]$ [P, p.4] of G over H^*, in which the multiplication is twisted by the 2-cocycle $\theta^*\colon G \times G \to H^*$. As easily seen, a twisted group ring of a cyclic group over a commutative ring is commutative. Hence for the non-commutativity of A^*, G must not be cyclic. Hence Part (2) follows.

(3) By the same reason as in (2), $\rightharpoonup$ must not be trivial, since A is isomorphic to the crossed product $K * C_2$ [P, p.2] with data $\rightharpoonup, \sigma$. To complete the proof, we claim the action of z is a Hopf algebra automorphism. In fact, this follows, since z acts on $K \cong k^G$ as an automorphism compatible with counit ε_K and since the square of the action is identical by [P, Lemma 1.1 ii]. (This follows alternatively, since (H, K) together with $\rightharpoonup, \rho$ forms an **abelian matched pair** [T, Def. 1.1]. See [Si, Props, 2.5, 2.6], [H1, Satz 3.1.13; H2, Prop. 3.8].) ∎

Next we choose suitable θ. We use the notation in (2.3.3). Thus $G = \{1, x, y, xy\}$. We regard $K = k\langle x\rangle \otimes k\langle y\rangle = k^{\langle x\rangle} \otimes k^{\langle y\rangle} = K^*$, so that the dual basis e_{ij} of $x^i y^j$ is given by

$$e_{ij} = \frac{1}{4}(1 + (-1)^i x)(1 + (-1)^j y) \quad (i, j = 0, 1)$$

(see the paragraph just above (1.6)).

PROPOSITION 2.4: *By a change of the isomorphism α, θ is chosen so that*

$$\theta(z) = \sum_{0 \le i,j,k,l \le 1} (-1)^{jk} e_{ij} \otimes e_{kl}.$$

(Note θ is unitary, since α is unitary.)

Proof: As remarked in the proof of (2.3.2), $A^* \cong H^{*t}[G]$ via α^*. A change of α^* corresponds to a **diagonal change of basis** in $H^{*t}[G]$ [P, p. 3]. Note $H^* = k^{C_2} = ke_0 \times ke_1$, where e_i is the dual basis of z^i $(i = 0, 1)$. Since k is algebraically closed, there are units $\bar{x}, \bar{y}$ of $H^{*t}[G]$ in the x-, y-component such that $\bar{x}^2 = \bar{y}^2 = 1$. Set $u = \bar{y}^{-1}\bar{x}^{-1}\bar{y}\bar{x}$. Then $u \in H^*$ and $u^2 = 1$, since $\bar{y}u^2 = \bar{x}\bar{x}\bar{y}u^2 = \bar{x}\bar{y}\bar{x}u = \bar{x}\bar{y}u\bar{x} = \bar{y}\bar{x}\bar{x} = \bar{y}$. For the non-commutativity of A^*, u must not be 1. Furthermore $u(1) = 1$, since α^* is co-unitary. Hence $u = e_0 - e_1$. The 2-cocycle with respect to the basis $1, \bar{x}, \bar{y}, \bar{x}\bar{y}$ is given by

$$\theta^*(x^iy^j, x^ky^l) = e_0 + (-1)^{jk}e_1 \qquad (0 \le i, j, k, l \le 1).$$

Hence the Proposition follows. ∎

Finally to show the existence of our A, we find such cocycles σ that make $K \bowtie H$ a Hopf algebra in fact. Set

$$v = \sigma(z, z),$$

a unit in K (note σ is determined by v, since σ is normalized). It holds by the cocycle condition [P, Lemma 1.1 i] that

$$z \rightharpoonup v = v, \tag{2.5}$$

where $\rightharpoonup$ is as given in (2.3.3). Conversely, if v is unit in K satisfying (2.5), $K \bowtie H$ with $\rightharpoonup$, σ is at least a crossed product of H over K, in particular a K-ring, that is, an algebra given an algebra map $K \to K \bowtie H$. As a K-ring, $K \bowtie H$ is generated by $z = 1 \bowtie z$ with relations

$$z^2 = v, \qquad zc = (z \rightharpoonup c)z \qquad (c \in K). \tag{2.6}$$

On the other hand, $K \bowtie H$ is at least a coalgebra with structure Δ, ε such that

$$\Delta(z) = \theta(z)(z \otimes z), \qquad \varepsilon(z) = 1, \tag{2.7}$$

where θ is as given in (2.4), and that the following diagrams commute:

$$\begin{array}{ccc} K \bowtie H & \xrightarrow{\Delta} & (K \bowtie H) \otimes (K \bowtie H) \\ \uparrow & & \uparrow \\ K & \xrightarrow{\Delta_K} & K \otimes K \end{array} \qquad \begin{array}{ccc} K \bowtie H & \xrightarrow{\varepsilon} & k \\ \uparrow & & \Vert \\ K & \xrightarrow{\varepsilon_K} & k \end{array}$$

PROPOSITION 2.8: *σ together with $\rightharpoonup, \rho, \theta$ as given in* (2.3), (2.4) *makes $K \bowtie H$ a Hopf algebra, if and only if v equals*

$$\text{either} \quad v_1 = \frac{1}{2}(1 + x + y - xy) \quad \text{or} \quad v_2 = \frac{1}{2}(-1 + x + y + xy).$$

Proof: Write

$$v = \sum_{0 \le i,j \le 1} c_{ij} e_{ij},$$

where $c_{ij} \in k$. For v to be a unit satisfying (2.5), it should hold that

$$c_{ij} \neq 0, \qquad c_{10} = c_{01}. \tag{2.9}$$

Then $K \bowtie H$ is a K-ring as cited above. Under (2.9), σ makes $K \bowtie H$ a bialgebra, if and only if the K-ring maps Δ, ε determined by (2.7) are well-defined, where $(K \bowtie H) \otimes (K \bowtie H), k$ are regarded as K-rings via Δ_K, ε_k. This is equivalent to the following four conditions:

$$\begin{aligned} &\theta(z)(z \otimes z)\theta(z)(z \otimes z) = \Delta_K(v), \\ &\varepsilon(z)^2 = \varepsilon_K(v), \\ &\theta(z)(z \otimes z)\Delta_K(c) = \Delta_K(z \rightharpoonup c)\theta(z)(z \otimes z), \\ &\varepsilon(z)\varepsilon_K(c) = \varepsilon_K(z \rightharpoonup c)\varepsilon(z) \quad (c \in K). \end{aligned}$$

The 3rd and 4th conditions hold automatically, since z acts on K as a coalgebra automorphism. The 1st and 2nd conditions hold, if and if only, respectively,

$$\begin{cases} (1-)^{il+jk} c_{ij} c_{kl} = c_{pq}, \\ \text{whenever } i + k \equiv p, \quad j + l \equiv q \pmod 2; \end{cases} \tag{2.10}$$

$$c_{00} = 1. \tag{2.11}$$

We solve (2.9)–(2.11) to have

$$c_{00} = 1, \quad c_{10} = c_{01} = \pm 1, \quad c_{11} = -1.$$

Describing v by means of x, y, we have v_1, v_2 in the Proposition.

It is shown in [H1, Satz 5.1; H2, Prop. 3.13] that a bialgebra which is the bicrossed product of two Hopf algebras is a Hopf algebra. (For an explicit description of the antipode, see (2.13) below.) Hence the Proposition follows. ∎

Denote by $\mathcal{A}$ (resp. $\mathcal{A}'$) the Hopf algebra obtained by taking v_1 (resp. v_2) together with $\rightharpoonup, \rho, \theta$ as in (2.8). These are semisimple, since by [DT, Thm. 3.14] a crossed product of a semisimple Hopf algebra over a semisimple algebra is semisimple.

LEMMA 2.12: *$\mathcal{A} \cong \mathcal{A}'$ as Hopf algebras.*

Proof: The K-ring map $z \mapsto yz$, $\mathcal{A} \to \mathcal{A}'$ gives a Hopf algebra isomorphism. ∎

Thus we obtain the following:

THEOREM 2.13: *8-Dimensional semisimple Hopf algebras over an algebraically closed field k of characteristic $\neq 2$ consist of 8 isomorphic classes, which are represented by*

$$k(C_2 \times C_2 \times C_2),\quad k(C_2 \times C_4),\quad kC_8,$$
$$kD,\quad k^D,\quad kQ,\quad k^Q,\quad \mathcal{A},$$

where $D = C_4 \rtimes C_2$ is the dihedral group and Q is the quaternion group. Among these, $\mathcal{A}$ is the unique one that is neither commutative nor cocommutative, and is generated as an algebra by x, y, z with relations

$$x^2 = y^2 = 1, \quad z^2 = \frac{1}{2}(1 + x + y - xy), \quad yx = xy, \quad zx = yz, \quad zy = xz;$$

the coalgebra structure Δ, ε and the antipode S are determined by

$$\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y, \quad \varepsilon(x) = \varepsilon(y) = 1,$$
$$\Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z), \quad \varepsilon(z) = 1,$$
$$S(x) = x, \quad S(y) = y, \quad S(z) = z.$$

(We remark that S is an algebra anti-automorphism and that $S \neq 1, S^2 = 1$.)

REMARK 2.14: Suppose k is a field of characteristic $\neq 2$ which is not necessarily algebraically closed.

(1) $\mathcal{A}$ is (2.13) can be defined over k. $\mathcal{A}^*$ is presented as follows: $\mathcal{A}^*$ is generated by c, s, h with relations

$$c^2 - s^2 = 1, \quad sc = cs = 0, \quad h^2 = 1, \quad ch = hc, \quad sh = -hs,$$

given the structures determined by

$$\Delta(c) = c \otimes c - s \otimes s, \quad \varepsilon(c) = 1,$$
$$\Delta(s) = c \otimes s + s \otimes c, \quad \varepsilon(s) = 0,$$
$$\Delta(h) = h \otimes h + hs^2 \otimes h(1 - c - s), \quad \varepsilon(h) = 1,$$
$$S(c) = c, \quad S(s) = s, \quad S(h) = h(s^2 + s + 1).$$

(2) Suppose k contains an 8th primitive root ζ of 1 , and write $\sqrt{-1} = \zeta^2$, a square root of -1. Set

$$w = (e_{00} + \zeta e_{10} + \zeta^{-1} e_{01} + \sqrt{-1} e_{11}) z.$$

Then $\mathcal{A}$ is presented by generators x, y, w, where the expressions containing z are replaced by

$$w^2 = 1, \quad wx = yw,$$

$$\Delta(w) = \left(\frac{1}{2}(1+xy) \otimes 1 + \tfrac{1+\sqrt{-1}}{4}(1-xy) \otimes x + \tfrac{1-\sqrt{-1}}{4}(1-xy) \otimes y \right)(w \otimes w),$$

$$S(w) = \left(\tfrac{1+\sqrt{-1}}{2} x + \tfrac{1-\sqrt{-1}}{2} y \right) w.$$

(3) $\mathcal{A}^*$ is semisimple, and neither commutative nor cocommutative. Hence if k is algebraically closed, it should hold by (2.13) that $\mathcal{A} \cong \mathcal{A}^*$. In fact

$$x \mapsto c + \sqrt{-1} s, \quad y \mapsto c - \sqrt{-1} s, \quad w \mapsto h$$

gives an isomorphism. But if $\sqrt{-1} \notin k$, these are not isomorphic, since the group-likes $G(\mathcal{A}^*)$ in $\mathcal{A}^*$ consist of two elements $1, c^2 + s^2$.

Proof: (1) We know $\mathcal{A}^* = K^* \bowtie H^*$ with data dual to $\mathcal{A}$'s, where $K^* = k\langle x, y\rangle$, $H^* = k\langle z\rangle$. Therefore $\mathcal{A}^*$ is presented by generators x, y, z with relations

$$x^2 = y^2 = z^2 = 1, \quad zx = xz, \quad zy = yz, \quad yx = xyz$$

given the structures

$$\Delta(x) = xe_0 \otimes x + xe_1 \otimes y, \quad \varepsilon(x) = 1,$$

$$\Delta(y) = ye_0 \otimes y + ye_1 \otimes x, \quad \varepsilon(y) = 1,$$

$$\Delta(z) = z \otimes z, \quad \varepsilon(z) = 1,$$

$$S(x) = xe_0 + ye_1, \quad S(y) = ye_0 + xe_1, \quad S(z) = z,$$

where e_i is as given in (1.5). Set

$$c = xye_0, \quad s = xye_1, \quad h = x.$$

Then Part (1) follows.

(2), (3) Straightforward.

Note added in revision. Reading the original version of this paper, Dr. T. Masuda and Dr. Y. Sekine informed me that G.I. Kac had discovered in the 1960's a non-commutative, non-cocommutative semisimple Hopf algebra of dimension 8. It follows by the uniqueness in Theorem 2.13 that this Hopf algebra of Kac's is isomorphic to our $\mathcal{A}$.

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